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Hypersonic weak-interaction similarity solutions for flow past a flat plate

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The hypersonic weak-interaction regime for the flow of a viscous, heat-conducting compressible fluid past a flat plate is analysed using the Navier–Stokes equations as a basis. The fluid is assumed to be a perfect gas having constant specific heats, a constant Prandtl number, σ, of order unity, and a viscosity coefficient varying as a power, ω, of the absolute temperature. Limiting forms of solutions are studied for the free-stream Mach number, M, the free-stream Reynolds number (based on the plate length), R_L, and the reciprocal of the weak-interaction parameter, $(\chi^*)^{-1} = \mathscr{F}(M, R_L, \omega, \sigma)$, greater than order unity.

By means of matched asymptotic expansions, it is shown that, for $(1-\omega) > 0$, the zone between the shock wave and the plate is composed of four distinct regions for which similarity exists. The behaviour of the flow in these four regions is analysed.

1. Introduction

The proper formulation of the hypersonic weak-interaction theory (HWIT) for viscous compressible flow past a flat plate has been a subject of considerable interest over the past decade (compare, for example, Kuo 1956; Hayes & Probstein 1959*a*, *b*; Freeman & Lam 1959*a*, *b*; Stewartson 1964). The purpose of this paper is to review and enlarge upon the existing formulations of the HWIT flat plate problem, and, in the process, present the uniformly valid self-similar solutions for the distinct physical regions which characterize the entire HWIT flow field. The presentation is carried out along lines corresponding to those employed by Bush (1966) in analysing the hypersonic strong-interaction theory (HSIT) flat plate problem.

In § 2, the von Mises forms of the Navier–Stokes equations of motion are given.

In § 3, the analysis of the primary inviscid shock layer, supported by a viscous boundary layer with a thickness ratio of $O(\delta)$, is presented. This analysis, which is just the linearized supersonic flow theory in a modified hypersonic form, closely follows those given by Kuo (1956), for his high supersonic flow theory, and Stewartson (1964), for his HWIT.

The linearized theory of § 3 fails to give the characteristic slope with sufficient accuracy for the shock wave to be accommodated within the primary inviscid shock layer formulation. The non-uniformity exhibited by the linearized theory in the vicinity of the shock wave has long been recognized, with the generally accepted technique for the removal of this non-uniformity being the application of the co-ordinate straining method of Lighthill (1949) (cf. Van Dyke 1964).

Rather than modify the primary inviscid layer analysis through the introduction of the method of Lighthill, the present authors, in §4, introduce an additional inviscid layer, referred to as the exterior inviscid layer, adjacent to the shock wave, which removes the non-uniformity. This formulation follows that of Cole (1966) in his treatment of non-steady one-dimensional gasdynamics problems. It shows explicitly the difference of the orders of magnitude of the flow quantities near the shock front from those in the primary inviscid layer, and, hence, the authors believe, provides an important amplification and clarification of the flow field picture near the shock as presently provided by the co-ordinate straining method.

In §5, the HWIT viscous boundary layer formulation is presented in a form which parallels that of the HSIT viscous boundary layer formulation of Bush (1966). The solutions of the equations for this layer are not presented, but, rather, the asymptotic behaviours of the flow quantities near the outer edge of the layer are given. The asymptotic behaviours for the case of $\mu \sim T^{\omega}$, $\omega < 1$ are given in detail, while those for $\mu \sim T$ are noted.

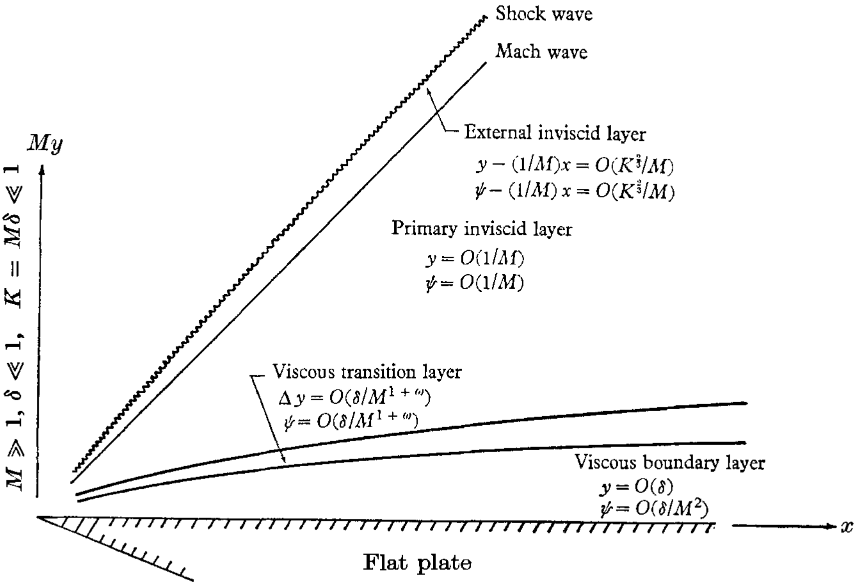

FIGURE 1. Schematic diagram of hypersonic weak-interaction layers for flow past a flat plate.

From these asymptotic behaviours, it is seen that the solutions for the viscous boundary layer do not match directly to the solutions for the primary inviscid layer. In §6, a viscous transition layer formulation is presented for the case of $\omega < 1$. It is shown that the solutions of this viscous transition layer match directly to both the solutions of the primary inviscid shock layer and those of the viscous boundary layer, and, thus, complete a uniformly valid picture of the flow field from the plate to the shock wave.

Hayes & Probstein (1959*b*) and Freeman & Lam (1959*a*, *b*) have indicated that, for this weak-interaction problem, there must be a (viscous) transition layer at the outer edge of the hypersonic viscous boundary layer in order to accommodate the large temperature difference between the hot viscous boundary layer and the (relatively) cold inviscid shock layer. However, in these 1959 papers, the need for the transition layer is presented from the viewpoint of obtaining a correction to the viscous boundary layer rather than from the viewpoint of obtaining the picture of the entire flow field (extending from the plate out to the shock wave).

As in the case of HSIT, the finding of the HWIT solutions for $\mu \sim T$ (i.e. $\omega = 1$) represents a special case, one which merits further investigation.

2. The equations of motion†

Consider the (two-dimensional) flow of a viscous, compressible gas past a semi-infinite flat plate. Let $x_1 = Lx$ and $y_1 = Ly$ represent the Cartesian co-ordinates parallel and normal to the flat plate, respectively, with the origin of this co-ordinate system at the leading edge of the plate. The length L is chosen so that x is of order unity in the region where the weak-interaction theory is valid. The velocity components in the x_1- and y_1-directions are $u_1 = u_\infty u$, and $v_1 = u_\infty v$, and the pressure, temperature, and density are $p_1 = p_\infty p$, $T_1 = T_\infty T$, and $\rho_1 = \rho_\infty \rho$, where u_∞, p_∞, T_∞, and ρ_∞ are the velocity in the x_1-direction, pressure, temperature, and density in the undisturbed region upstream of the flat plate.

The gas is assumed to be a perfect one ($p = \rho T$), having (i) constant specific heats, c_{v_1} and c_{p_1}, with $\gamma = (c_{p_1}/c_{v_1})$, such that $(\gamma - 1) = O(1)$, (ii) a constant Prandtl number of order unity ($\sigma = \text{const.} = O(1)$), and (iii) its 'normal' viscosity coefficient proportional to a power, ω, of the absolute temperature

$$(\mu_1 = \mu_\infty \mu = \mu_\infty T^\omega, \quad \text{with} \quad \tfrac{1}{2} \leqslant \omega < 1,$$

as will be shown to be required in the succeeding analysis), while its 'bulk' viscosity coefficient is taken to be zero, although such an assumption is not necessary.

The von Mises forms of the Navier–Stokes equations for the flow of such a gas are

$$\frac{\partial}{\partial \psi}\left(\frac{v}{u}\right) - \frac{\partial}{\partial \xi}\left(\frac{1}{\rho u}\right) = 0, \tag{2.1}$$

$$\begin{aligned} \rho u \frac{\partial u}{\partial \xi} &+ \frac{1}{\gamma M^2}\left(\frac{\partial p}{\partial \xi} - \rho v \frac{\partial p}{\partial \psi}\right) \\ &= \frac{1}{R_L}\left[\!\left[\rho u \frac{\partial}{\partial \psi}\left[T^\omega \left\{\rho u \frac{\partial u}{\partial \psi} + \left(\frac{\partial v}{\partial \xi} - \rho v \frac{\partial v}{\partial \psi}\right)\right\}\right] + \left(\frac{\partial}{\partial \xi} - \rho v \frac{\partial}{\partial \psi}\right)\right.\right. \\ &\qquad \left.\left. \times \left[T^\omega \left\{\frac{4}{3}\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right) - \tfrac{2}{3}\rho u \frac{\partial v}{\partial \psi}\right\}\right]\right]\!\right], \end{aligned} \tag{2.2}$$

$$\begin{aligned} \rho u &\left(\frac{\partial v}{\partial \xi} + \frac{1}{\gamma M^2}\frac{\partial p}{\partial \psi}\right) \\ &= \frac{1}{R_L}\left[\!\left[\rho u \frac{\partial}{\partial \psi}\left[T^\omega \left\{\tfrac{4}{3}\rho u \frac{\partial v}{\partial \psi} - \frac{2}{3}\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right)\right\}\right] + \left(\frac{\partial}{\partial \xi} - \rho v \frac{\partial}{\partial \psi}\right)\right.\right. \\ &\qquad \left.\left. \times \left[T^\omega \left\{\rho u \frac{\partial u}{\partial \psi} + \left(\frac{\partial v}{\partial \xi} - \rho v \frac{\partial v}{\partial \psi}\right)\right\}\right]\right]\!\right], \end{aligned} \tag{2.3}$$

† To emphasize the parallelism of the present hypersonic weak-interaction theory with the hypersonic strong-interaction theory of Bush (1966), the equations of motion of both theories are given in the identical form.

$$\rho u \frac{\partial T}{\partial \xi} - \left(\frac{\gamma-1}{\gamma}\right) u \frac{\partial p}{\partial \xi}$$
$$= \frac{1}{\sigma R_L}\left[\left[\rho u \frac{\partial}{\partial \psi}\left(T^{\omega}\rho u \frac{\partial T}{\partial \psi}\right) + \left(\frac{\partial}{\partial \xi} - \rho v \frac{\partial}{\partial \psi}\right)\left(T^{\omega}\left\{\frac{\partial T}{\partial \xi} - \rho v \frac{\partial T}{\partial \psi}\right\}\right)\right]\right]$$
$$+ \left(\frac{\gamma-1}{\gamma}\right)\frac{\gamma M^2}{R_L} T^{\omega}\left[\left[\left\{\rho u \frac{\partial u}{\partial \psi} + \left(\frac{\partial v}{\partial \xi} - \rho v \frac{\partial v}{\partial \psi}\right)\right\}^2 + 2\left\{\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right)^2 + \left(\rho u \frac{\partial v}{\partial \psi}\right)^2\right\}\right.\right.$$
$$\left.\left. - \frac{2}{3}\left\{\left(\frac{\partial u}{\partial \xi} - \rho v \frac{\partial u}{\partial \psi}\right) + \rho u \frac{\partial v}{\partial \psi}\right\}^2\right]\right], \qquad (2.4)$$

where $\xi = x$ and ψ is the stream function, defined by

$$(\partial \psi/\partial y) = \rho u, \quad (\partial \psi/\partial x) = -\rho v;$$

while $M^2 = (\rho_\infty u_\infty^2/\gamma p_\infty)$, the square of the free-stream Mach number; and $R_L = (\rho_\infty u_\infty L/\mu_\infty)$, the Reynolds number. The analysis presented here is for $M^2 \gg 1$ and $R_L \gg 1$.

3. The primary inviscid (shock) layer

According to the existing hypersonic weak-interaction theory for flow past a flat plate, at the surface there is a thin, viscous, heat-conducting layer, which disturbs the external flow. This layer, whose outer edge is given by $y = \delta Y_k(x) + \ldots$, with δ, the layer's thickness parameter, much less than unity, acts as an effective slender 'body', producing a weak oblique (Rankine–Hugoniot) shock wave, $y = (1/M)x + \ldots$, for $K = M\delta \ll 1$, and an inviscid shock layer between the shock wave and the 'body'. The analysis of this primary inviscid layer provides the starting point for this paper.

For such an inviscid layer, the flow quantities have the following representations (cf. Stewartson 1964; Hayes & Probstein 1959*a*):

$$\xi_a = \xi, \quad \psi_a = M\psi; \qquad (3.1)$$

$$\left.\begin{aligned} u &= 1 + (\delta/M)u_a + \ldots = 1 + (K/M^2)u_a + \ldots, \\ v &= \delta v_a + \ldots = (K/M)v_a + \ldots, \\ p &= 1 + (M\delta)p_a + \ldots = 1 + Kp_a + \ldots, \\ T &= 1 + (M\delta)T_a + \ldots = 1 + KT_a + \ldots, \\ \rho &= 1 + (M\delta)\rho_a + \ldots = 1 + K\rho_a + \ldots, \end{aligned}\right\} \qquad (3.2)$$

where $f_a = f_a(\xi_a, \psi_a) = O(1)$.

Substitution of (3.1) and (3.2) into the equations of motion yields, to first approximation,

$$\frac{\partial v_a}{\partial \psi_a} + \frac{\partial \rho_a}{\partial \xi_a} = 0, \quad p_a - (\rho_a + T_a) = 0,$$

$$\frac{\partial}{\partial \xi_a}\left(u_a + \frac{1}{\gamma}p_a\right) = 0, \quad \frac{\partial v_a}{\partial \xi_a} + \frac{1}{\gamma}\frac{\partial p_a}{\partial \psi_a} = 0, \quad \frac{\partial}{\partial \xi_a}\left(T_a - \frac{\gamma-1}{\gamma}p_a\right) = 0, \qquad (3.3)$$

where the ratio of the orders of magnitude of the leading viscosity and heat-conduction terms, which have been neglected, to those of the inviscid terms, which have been retained, is

$$(M^2/R_L) = (M^{2(1+\omega)}/R_L\delta^2)(\delta/M^{\omega})^2.$$

With rearrangement, (3.3) becomes

$$\frac{\partial^2 v_a}{\partial \xi_a^2} - \frac{\partial^2 v_a}{\partial \psi_a^2} = 0, \quad \frac{\partial p_a}{\partial \psi_a} = -\gamma \frac{\partial v_a}{\partial \xi_a},$$

$$u_a = -\frac{1}{\gamma} p_a + U_a(\psi_a), \quad T_a = \frac{\gamma - 1}{\gamma} p_a + H_a(\psi_a), \quad \rho_a = \frac{1}{\gamma} p_a - H_a(\psi_a). \tag{3.4}$$

With the 'body' boundary condition,

$$v_a(\xi_a, 0) = Y_k'(\xi_a), \tag{3.5}$$

the solutions of (3.4) (cf. Kuo (1956) and Stewartson (1964)) are the following outgoing simple wave solutions, with $\tau_a = (\xi_a - \psi_a)$:

$$\left.\begin{aligned} &v_a(\xi_a, \psi_a) = Y_k'(\tau_a) = V_a(\tau_a), \quad p_a(\xi_a, \psi_a) = \gamma V_a(\tau_a), \\ &u_a(\xi_a, \psi_a) = -V_a(\tau_a) + U_a(\psi_a), \quad T_a(\xi_a, \psi_a) = (\gamma - 1) V_a(\tau_a) + H_a(\psi_a), \\ &\rho_a(\xi_a, \psi_a) = V_a(\tau_a) - H_a(\psi_a). \end{aligned}\right\} \tag{3.6}$$

For the case of $Y_k(\xi) = 2A_k \xi^{\frac{1}{2}}$ (so that $Y_k'(\xi) = A_k \xi^{-\frac{1}{2}}$), which, in §5, is shown to be a requirement for a self-similar flow in the viscous boundary layer, (3.6) reduces to

$$\begin{aligned} v_a = p_a/\gamma &= -u_a + U_a(\psi_a) = (T_a - H_a(\psi_a))/(\gamma - 1) = \rho_a + H_a(\psi_a) \\ &= A_k \tau_a^{-\frac{1}{2}} \quad (A_k = \text{const.}). \end{aligned} \tag{3.7†}$$

To evaluate the functions $U_a(\psi_a)$ and $H_a(\psi_a)$, (3.6) and/or (3.7) must be studied in the vicinity of the shock wave, where $\tau_a \to 0$, i.e. $\psi_a \to \xi_a$. However, it can be shown that the solutions, v_a, p_a, etc., exhibit a singular behaviour as $\tau_a \to 0$. Hence, in order to remove this singularity and develop a solution uniformly valid from the shock wave to the plate, a thin exterior inviscid layer is introduced spanning the distance between the primary inviscid layer and the Rankine–Hugoniot shock.

4. The exterior inviscid layer

The weak-interaction shock relations are now considered. The shock wave in the weak-interaction limit ($M \to \infty$, $M\delta = K \to 0$) corresponds to a small disturbance (of a magnitude to be determined) on a Mach wave. Thus, in this limit, if the shock shape and the shock stream function are given by

$$\left.\begin{aligned} y_{sh}(x) &= (1/M)[x + K_f F(x) + \ldots], \\ \psi_{sh}(\xi) &= (1/M)[\xi + K_f F(\xi) + \ldots], \\ K_f &= \text{undetermined parameter} \ll 1, \end{aligned}\right\} \tag{4.1}$$

then the flow quantities at the downstream side of the shock are

$$\begin{aligned} (1 - u_{sh})/(K_f/M^2) &= v_{sh}/(K_f/M) = (p_{sh} - 1)/\gamma K_f = (T_{sh} - 1)/(\gamma - 1) K_f \\ &= (\rho_{sh} - 1)/K_f = \{4/(\gamma + 1)\} F'(\xi) + \ldots. \end{aligned} \tag{4.2}$$

(Note: $(p/\rho^\gamma)_{sh} = 1 + O(K_f^3)$, $\tau_{sh} = \{\xi - M\psi_{sh}(\xi)\} = -K_f F(\xi) + \ldots$.)

† The solutions of (3.7), for $U_a = H_a = 0$, may be considered to be self-similar solutions of the form

$$v_a = p_a/\gamma = -u_a = T_a/(\gamma - 1) = \rho_a = A_k \xi_a^{-\frac{1}{2}} (1 - \zeta_a)^{-\frac{1}{2}}, \quad \zeta_a = (\psi_a/\xi_a).$$

The forms of the solutions of (3.6) and (3.7) and the shock relations of (4.1) and (4.2) suggest that the proper representations for the flow quantities in this layer adjacent to the shock front are

$$\psi_f = M\psi, \quad \tau_f = (\xi - M\psi)/K_f, \tag{4.3}$$

$$\left.\begin{aligned} u = 1+(K_f/M^2)\,u_f+\ldots, \quad v = (K_f/M)\,v_f+\ldots, \quad p = 1+K_f p_f+\ldots, \\ T = 1+K_f T_f+\ldots, \quad \rho = 1+K_f\rho_f+\ldots, \end{aligned}\right\} \tag{4.4}$$

where $f_f = f_f(\psi_f, \tau_f) = O(1)$. The shock relations for these representations are

$$(v_f)_{sh} = -(u_f)_{sh} = (\rho_f)_{sh} = (p_f)_{sh}/\gamma = (T_f)_{sh}/(\gamma-1) = \{4/(\gamma+1)\}F'(\xi), \tag{4.5}$$

where $(f_f)_{sh} = f_f(\xi, -F(\xi))$.

Following Cole (1966)†, a 'Riemann invariant' formulation of the equations of motion for this layer yields:

$$\left.\begin{aligned} &\frac{\partial v_f}{\partial \psi_f} + \frac{1}{\gamma}\left(\frac{\partial p_f}{\partial \psi_f} - \frac{\gamma+1}{\gamma} p_f \frac{\partial p_f}{\partial \tau_f}\right) = 0, \quad p_f - (\rho_f + T_f) = 0, \\ &\frac{\partial}{\partial \tau_f}\left(u_f + \frac{1}{\gamma} p_f\right) = 0, \quad \frac{\partial}{\partial \tau_f}\left(v_f - \frac{1}{\gamma} p_f\right) = 0, \quad p_f - \gamma\rho_f = 0. \end{aligned}\right\} \tag{4.6}$$

Applying the shock relations, (4.5) to (4.6), it follows that

$$p_f/\gamma = \rho_f = -u_f = T_f/(\gamma-1) = v_f, \tag{4.7}$$

where the governing equation for v_f is

$$\frac{\partial v_f}{\partial \psi_f} - \left(\frac{\gamma+1}{2}\right) v_f \frac{\partial v_f}{\partial \tau_f} = 0. \tag{4.8}$$

The general solution of (4.8) is

$$\tau_f = G_f(v_f) - \tfrac{1}{2}(\gamma+1)\,\psi_f v_f.$$

However, the assumption of a self-similar solution, with the variables

$$v_f = \psi_f^{m-1} V_f(\zeta_f), \quad \zeta_f = (\tau_f/\psi_f^m), \tag{4.9}$$

reduces (4.8) to the ordinary differential equation

$$[\tfrac{1}{2}(\gamma+1)V_f + m\zeta_f]\frac{dV_f}{d\zeta_f} + (1-m)V_f = 0. \tag{4.10}$$

The solution of this equation, in terms of the original variables, is

$$\tau_f = \left(\frac{J_f}{v_f^{m/(1-m)}}\right) - \tfrac{1}{2}(\gamma+1)\,\psi_f v_f, \quad J_f = \text{const}. \tag{4.11}$$

(Note that the first term in (4.11) represents the restricted similarity form of the function $G_f(v_f)$ in the general solution of (4.8).)

That the solutions of (4.7), as defined by (4.11), can be matched to the special solutions, $\tau_a = (A_k/v_a)^2$, etc., of (3.7) (thus removing the singular behaviour of the primary inviscid layer solutions near the shock wave) is now demonstrated.

† Cole's work is carried out for the non-steady one-dimensional inviscid flow problem and applies to the present analysis by virtue of the 'law of plane sections'.

Let $v = (K/M)v_a + \ldots = (K_f/M)v_f + \ldots = \beta_{af}v_{af} + \ldots,$

where $v_{af} = O(1), \quad \text{and} \quad (K/M) \ll \beta_{af} \ll (K_f/M) \ll 1,$

so that the matching of the two sets of solutions takes place as $v_a \to \infty$, and $v_f \to 0$, $\psi_a = \psi_f =$ fixed. For the primary inviscid layer, then,

$$\tau = \xi - M\psi = \xi_a - \psi_a = \tau_a = A_k^2(K/M\beta_{af}v_{af})^2 + \ldots, \tag{4.12a}$$

while, for the exterior inviscid layer,

$$\tau = \xi - M\psi = K_f\tau_f = K_f[J_f(K_f/M\beta_{af}v_{af})^{m/(1-m)} - \tfrac{1}{2}(\gamma+1)\psi_f(M\beta_{af}v_{af}/K_f) + \ldots]. \tag{4.12b}$$

Examination of (4.12) reveals that the matching is accomplished if

$$m = \tfrac{2}{3}, \quad K_f = K^{\frac{2}{3}} \gg K, \quad J_f = A_k^{\frac{2}{3}}. \tag{4.13}$$

A corollary of the above matching is that $U_a(\psi_a) = H_a(\psi_a) = 0$.

Evaluating (4.11) at the shock, by means of (4.5) and (4.13), yields the following equation to be satisfied by the perturbation shock shape, $F(\xi)$:

$$-F(\xi) = \{(\gamma+1)A_k/4\}^2\{F'(\xi)\}^{-2} - 2\xi F'(\xi). \tag{4.14}$$

The solution of this equation is

$$F(\xi) = \tfrac{3}{4}\{(\gamma+1)A_k\}^{\frac{2}{3}}\xi^{\frac{2}{3}}. \tag{4.15}$$

Hence, the shock shape, consistent with a viscous boundary layer whose outer edge is given by $y = \delta(2A_k x^{\frac{1}{2}}) + \ldots$, is

$$y_{sh}(x) = (1/M)[x + K^{\frac{2}{3}}\tfrac{3}{4}\{(\gamma+1)A_k\}^{\frac{2}{3}}x^{\frac{2}{3}} + \ldots]. \tag{4.16}$$

This result is compatible with that obtained by Kuo (1956). It has also been qualitatively verified by the experiments of Kendall (1957).

5. The viscous boundary layer

The viscous boundary layer (next to the plate), a high temperature, low density region, across which the pressure is constant, is examined next. From the solutions of the primary inviscid layer as $\psi_a \to 0$ given in § 3, at the outer edge of this viscous layer, it is expected that the flow quantities have the behaviour $u \to 1$, $T/M^2 \to 0$, $v \to \delta Y_k'(\xi)$, $p \to 1$.

The analysis of this region is carried out in the distorted co-ordinates

$$\xi_k = \xi, \quad \psi_k = (M^2/\delta)\psi, \tag{5.1}$$

and the expansions of the flow quantities have the form

$$\left.\begin{aligned} &u = u_k + \ldots, \quad v = \delta v_k + \ldots, \\ &T = (M^2)T_k + \ldots, \quad p = 1 + (M\delta)p_k + \ldots, \quad \rho = (1/M^2)(1/T_k) + \ldots, \end{aligned}\right\} \tag{5.2}$$

with $f_k = f_k(\xi_k, \psi_k) = O(1)$.

For these representations, the leading terms in the equations of motion are

$$\left.\begin{aligned} &\frac{\partial}{\partial\psi_k}\left(\frac{v_k}{u_k}\right) - \frac{\partial}{\partial\xi_k}\left(\frac{T_k}{u_k}\right) = 0, \quad \frac{\partial p_k}{\partial\psi_k} = 0, \\ &\frac{\partial u_k}{\partial\xi_k} = \left(\frac{M^{2(1+\omega)}}{R_L\delta^2}\right)\left[\frac{\partial}{\partial\psi_k}\left(\frac{u_k}{T_k^{1-\omega}}\frac{\partial u_k}{\partial\psi_k}\right)\right], \\ &\frac{\partial T_k}{\partial\xi_k} = \left(\frac{M^{2(1+\omega)}}{R_L\delta^2}\right)\left[\frac{1}{\sigma}\frac{\partial}{\partial\psi_k}\left(\frac{u_k}{T_k^{1-\omega}}\frac{\partial T_k}{\partial\psi_k}\right) + (\gamma-1)\frac{u_k}{T_k^{1-\omega}}\left(\frac{\partial u_k}{\partial\psi_k}\right)^2\right]. \end{aligned}\right\} \tag{5.3}$$

To retain the viscosity and heat-conduction terms, it is necessary that the quantity $(M^{2(1+\omega)}/R_L\delta^2) \equiv \lambda$ may be of order unity, so that

$$\delta = (M^{2(1+\omega)}/R_L\lambda)^{\frac{1}{2}} \to 0. \tag{5.4}$$

Combining (5.4) with the weak-interaction inequality $K = M\delta \ll 1$ yields

$$M^{2+\omega}/R_L^{\frac{1}{2}} \ll 1, \tag{5.5}$$

which is a generalization of the usual criterion for weak interaction, namely that the interaction parameter, $\chi = (M^3/R_L^{\frac{1}{2}})$ for $\omega = 1$, be much less than unity.

If the outer edge of the boundary layer is taken to be a power-law body of the form

$$y = \delta Y_k(x) + \ldots = \delta(A_k x^n/n) + \ldots,$$

so that $v_a(\xi_a, 0) = p_a(\xi_a, 0)/\gamma = A_k \xi_a^{-(1-n)}$, (5.3), subject to the boundary conditions at the outer edge,

$$u_k \to 1, \quad T_k \to 0, \quad v_k \to A_k \xi_k^{-(1-n)}, \quad p_k \to \gamma A_k \xi_k^{-(1-n)} \quad \text{as} \quad \psi_k \to \infty, \tag{5.6}$$

may be reduced to a system of ordinary differential equations if

$$n = \tfrac{1}{2}. \tag{5.7}$$

For $n = \frac{1}{2}$, taking the independent variables to be

$$\xi_k, \quad \zeta_k = \psi_k/\xi_k^{\frac{1}{2}}, \tag{5.8}$$

and taking the dependent variables to be

$$u_k = U_k(\zeta_k), \quad v_k = \xi_k^{-\frac{1}{2}} V_k(\zeta_k), \quad T_k = H_k(\zeta_k), \tag{5.9}$$

the continuity, momentum, and energy equations become

$$\left.\begin{aligned} 2\frac{d}{d\zeta_k}\left(\frac{V_k}{U_k}\right) + \zeta_k \frac{d}{d\zeta_k}\left(\frac{H_k}{U_k}\right) &= 0, \\ 2\lambda\left[\frac{d}{d\zeta_k}\left(\frac{U_k}{H_k^{1-\omega}}\frac{dU_k}{d\zeta_k}\right)\right] + \zeta_k\frac{dU_k}{d\zeta_k} &= 0, \\ 2\lambda\left[\frac{1}{\sigma}\frac{d}{d\zeta_k}\left(\frac{U_k}{H_k^{1-\omega}}\frac{dH_k}{d\zeta_k}\right) + (\gamma-1)\frac{U_k}{H_k^{1-\omega}}\left(\frac{dU_k}{d\zeta_k}\right)^2\right] + \zeta_k\frac{dH_k}{d\zeta_k} &= 0. \end{aligned}\right\} \tag{5.10}$$

The boundary conditions for these equations at the outer edge and at the wall, respectively, are

$$\begin{aligned} &U_k \to 1, \quad H_k \to 0, \quad V_k \to A_k, \quad \text{as} \quad \zeta_k \to \infty, \\ &U_k \to 0, \quad H_k \to H_{k,w} \neq 0, \quad V_k \to 0, \quad \text{as} \quad \zeta_k \to 0. \end{aligned} \tag{5.11}$$

(Note: from the continuity equation of (5.10) and the boundary conditions of (5.11), it can be seen that

$$A_k = \frac{1}{2}\int_0^\infty (H_k/U_k)\,d\zeta_k,$$

where U_k and H_k are determined from the coupled momentum and energy equations of (5.10).)

Apart from finding the complete solutions of the above equations (compare, for example, Dewey 1963), since $H_k \to 0$ and $U_k \to 1$ as $\zeta_k \to \infty$, consider the following asymptotic expansions for H_k and U_k as $\zeta_k \to \infty$:

$$\left.\begin{aligned} &H_k = C_k\zeta_k^{-r_k} + \ldots, \quad U_k = 1 + D_k\zeta_k^{-s_k} + \ldots, \\ &C_k, D_k = \text{consts.}, \quad r_k, s_k = \text{consts.} > 0. \end{aligned}\right\} \tag{5.12}$$

Substitution of these expansions into the momentum and energy equations of (5.10) produces the results that, for $(1-\omega) > 0$,

$$\left.\begin{aligned} r_k &= \frac{2}{1-\omega}, \quad C_k = \left[\left(\frac{2\lambda}{\sigma}\right)\left(\frac{1+\omega}{1-\omega}\right)\right]^{1/(1-\omega)}, \\ s_k &= \left[1+\frac{1}{\sigma}\left(\frac{1+\omega}{1-\omega}\right)\right] = r_k\left[1+\left(\frac{1-\sigma}{\sigma}\right)\left(\frac{1+\omega}{2}\right)\right], \quad D_k = \text{undetermined.} \end{aligned}\right\} \tag{5.13}$$

(Note: the coefficient D_k depends on the complete solution of the similarity boundary layer equations, and, in particular, upon the boundary conditions at the wall. For example, for the case of $\sigma = 1$, it is easily verified that

$$D_k = -2C_k/\{(\gamma-1)+2H_{k,w}\}.)$$

Therefore, near the outer edge of the boundary layer ($\psi_k \to \infty$, ξ_k fixed), T_k exhibits the behaviour

$$T_k = C_k \xi_k^{1/(1-\omega)} \psi_k^{-2/(1-\omega)} + \dots \quad [(1-\omega) > 0]. \tag{5.14}$$

Since this functional form for the temperature as $\psi_k \to \infty$ is incompatible with that given by (3.6) as $\psi_a \to 0$, it is evident that a transition layer must be introduced to ensure a uniformly valid solution for the temperature from the shock wave to the plate. Corresponding arguments apply to the solutions for the longitudinal velocity.

(Note: for $\omega = 1$, near the outer edge of the boundary layer, T_k exhibits the behaviour $\quad T_k = (\text{const.})\,(\psi_k/\xi_k^{\frac{1}{2}})^{-1}\exp\left(-\sigma\psi_k^2/4\lambda\xi_k\right) + \dots.$

Hence, for this case, the temperature goes to zero exponentially, rather than algebraically, as is true for $(1-\omega) > 0$, and there is no way to match directly to this exponential decay. Further discussion of this problem is given in Bush (1966).)

6. The viscous transition layer

A viscous transition layer, intermediate to the primary inviscid layer and the viscous boundary layer, is introduced to permit uniformly valid solutions for the temperature and longitudinal velocity. For this transition layer, the distorted co-ordinates and flow variable expansions are taken to be

$$\xi_t = \xi, \quad \psi_t = \psi/\phi_t, \quad \delta/M^2 \ll \phi_t \ll 1/M; \tag{6.1}$$

$$\left.\begin{aligned} u &= (1+\alpha_t u_t + \dots) + (\delta/M) w_t + \dots, \quad (\delta/M) \ll \alpha_t \ll 1, \\ v &= \delta v_t + \dots, \\ T &= (T_t + \dots) + (M\delta) S_t + \dots, \\ p &= 1 + (M\delta) p_t + \dots, \quad \rho = (1/T_t) + \dots, \end{aligned}\right\} \tag{6.2}$$

where $f_t = f_t(\xi_t, \psi_t) = O(1)$.

To match to the primary inviscid layer, it is necessary that

$$u_t \to 0, \quad T_t \to 1, \quad v_t \to A_k \xi_t^{-\frac{1}{2}}, \quad p_t \to \gamma A_k \xi_t^{-\frac{1}{2}} \quad \text{as} \quad \psi_t \to \infty. \tag{6.3a}$$

To match to the viscous boundary layer, it is necessary that

$$u_t \to \infty, \quad T_t \to \infty, \quad v_t \to A_k \xi_t^{-\frac{1}{2}}, \quad p_t \to \gamma A_k \xi_t^{-\frac{1}{2}} \quad \text{as} \quad \psi_t \to 0. \tag{6.3b}$$

The leading terms in the equations of motion, based on these co-ordinates and expansions, are, for $(\phi_t/\delta) \to 0$,

$$\left.\begin{aligned} \frac{\partial v_t}{\partial \psi_t} &= O\left(\frac{\phi_t}{\delta}\right) \to 0; \quad v_t = v_t(\xi_t) = A_k \xi_t^{-\frac{1}{2}}, \\ \frac{\partial p_t}{\partial \psi_t} &= O\left(K\frac{\phi_t}{\delta}\right) \to 0; \quad p_t = p_t(\xi_t) = \gamma A_k \xi_t^{-\frac{1}{2}}, \\ \frac{\partial u_t}{\partial \xi_t} &= \left[\left(\frac{M^{2(1+\omega)}}{R_L \delta^2}\right)\left(M^{1+\omega}\frac{\phi_t}{\delta}\right)^{-2}\right]\frac{\partial}{\partial \psi_t}\left(\frac{1}{T_t^{1-\omega}}\frac{\partial u_t}{\partial \psi_t}\right), \\ \frac{\partial T_t}{\partial \xi_t} &= \left[\left(\frac{M^{2(1+\omega)}}{R_L \delta^2}\right)\left(M^{1+\omega}\frac{\phi_t}{\delta}\right)^{-2}\right]\frac{1}{\sigma}\frac{\partial}{\partial \psi_t}\left(\frac{1}{T_t^{1-\omega}}\frac{\partial T_t}{\partial \psi_t}\right). \end{aligned}\right\} \tag{6.4}$$

Retention of the shear and heat-conduction terms requires that

$$\left(\frac{M^{2(1+\omega)}}{R_L \delta^2}\right)\left(M^{1+\omega}\frac{\phi_t}{\delta}\right)^{-2} = O(1). \tag{6.5}$$

That $(M^{2(1+\omega)}/R_L \delta^2) = \lambda = O(1)$ was required in §5. That $(M^{1+\omega}\phi_t/\delta) = O(1) \equiv 1$ remains to be demonstrated. However, if $(M^{1+\omega}\phi_t/\delta) = 1$, then it is true that $(\phi_t/\delta) = M^{-(1+\omega)} \to 0$, which was postulated in deriving (6.4).

The existence of self-similar solutions for the temperature and longitudinal velocity is now examined. If u_t and T_t have the forms

$$u_t = U_t(\zeta_t), \quad T_t = H_t(\zeta_t), \quad \text{where} \quad \zeta_t = \psi_t/\xi_t^{\frac{1}{2}}, \tag{6.6}$$

the longitudinal momentum and energy equations of (6.4) may be reduced to the following similarity forms:

$$\left.\begin{aligned} 2\lambda\frac{d}{d\zeta_t}\left(\frac{1}{H_t^{1-\omega}}\frac{dU_t}{d\zeta_t}\right) + \zeta_t\frac{dU_t}{d\zeta_t} &= 0, \\ \frac{2\lambda}{\sigma}\frac{d}{d\zeta_t}\left(\frac{1}{H_t^{1-\omega}}\frac{dH_t}{d\zeta_t}\right) + \zeta_t\frac{dH_t}{d\zeta_t} &= 0. \end{aligned}\right\} \tag{6.7}$$

Since H_t and $U_t \to \infty$ as $\zeta_t \to 0$, in order to match with the viscous boundary layer, consider the following asymptotic expansions:

$$\left.\begin{aligned} H_t = C_t \zeta_t^{-r_t} + \ldots, \quad U_t = D_t \zeta_t^{-s_t} + \ldots, \\ C_t, D_t = \text{consts.}, \quad r_t, s_t = \text{consts.} > 0. \end{aligned}\right\} \tag{6.8}$$

Substitution of these expansions into (6.7) produces the results that, for $(1-\omega) > 0$,

$$\left.\begin{aligned} r_t = r_k = 2/(1-\omega), \quad C_t = C_k = \left[\left(\frac{2\lambda}{\sigma}\right)\left(\frac{1+\omega}{1-\omega}\right)\right]^{1/(1-\omega)}, \\ s_t = s_k = \left[1 + \frac{1}{\sigma}\left(\frac{1+\omega}{1-\omega}\right)\right], \quad D_t = \text{undetermined.} \end{aligned}\right\} \tag{6.9}$$

From (5.12), (5.13) and (6.8), (6.9): (i) there is temperature matching at the transition layer/boundary layer 'interface' if

$$(M^{1+\omega}\phi_t/\delta) = 1, \quad \text{i.e. } \delta/M^2 \ll \phi_t = \delta/M^{1+\omega} \ll 1/M, \tag{6.10a}$$

which is exactly the relation that was required for the retention of the shear and heat-conduction terms in the transition layer equations; (ii) there is longitudinal velocity matching at this 'interface' if

$$D_t = D_k, \quad \alpha_t = M^{-2s_k/r_k}. \tag{6.10b}$$

The above value for α_t combined with the condition of (6.2) that α_t be much greater than (δ/M) leads to the requirement

$$K \ll M^2\alpha_t = M^{2\{1-(s_k/r_k)\}} = M^{-q}, \quad q = (1+\omega)\left(\frac{1-\sigma}{\sigma}\right). \tag{6.11}$$

For $\sigma = 1$, (6.11) reduces to the usual weak-interaction inequality, $K \ll 1$. However, for $\sigma < 1$, (6.11) becomes $K \ll M^{-q} \ll 1$, a more severe restriction than the above. Thus, for $\sigma \leqslant 1$, (5.5), the condition for hypersonic weak-interaction theory validity, must be modified to

$$\chi^* = (M^{\{1+[(1+\omega)/\sigma]\}}/R_L^{\frac{1}{2}}) \ll 1. \tag{6.12}$$

With respect to the transition layer/primary inviscid-layer 'interface', the solutions of (6.7), as $\zeta_t \to \infty$, are

$$\left.\begin{aligned} U_t &= (\text{const.})\,\text{erfc}\,(\zeta_t/2\lambda^{\frac{1}{2}}) + \ldots \to 0, \\ H_t &= 1 + (\text{const.})\,\text{erfc}\,(\sigma^{\frac{1}{2}}\zeta_t/2\lambda^{\frac{1}{2}}) + \ldots \to 1. \end{aligned}\right\} \tag{6.13}$$

Hence, the transition layer solutions join smoothly to those of the primary inviscid layer to the order considered.

(Note: the thickness of the transition layer is of $O(\phi_t) = O(\delta/M^{1+\omega}) \ll \delta$.)

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